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Hermite Functions on Compact Lie Groups, II

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The usual formula for Hermite polynomials on $\mathbf{R}^d$ is extended to a compact Lie group G, yielding an isometry of $L^2(G, p_1)$, where p_1 is the heat kernel measure at time one, with a natural completion of the universal enveloping algebra of G. The existence of such an isometry was first established by L. Gross; here we present a simplified proof as well as the explicit form of the isometry. © 1995 Academic Press, Inc.

Let $p_1(x)$ be the standard Gaussian on $\mathbf{R}^d$, i.e., the unique probability density on $\mathbf{R}^d$ with Fourier transform $e^{-|\xi|^2/2}$. Then it is well-known that the Hermite polynomials form an orthogonal basis for the complex Hilbert space $L^2(\mathbf{R}^d, p_1)$. More precisely let $S = S(\mathbf{C}^d)$ denote the vector space of symmetric tensors α over $\mathbf{C}^d$. Then there is a unique complex inner product on S such that the monomials form an orthogonal basis for S, each of length squared $(\mathrm{ord}(\alpha))!$, where $\mathrm{ord}(\alpha)$ is the total degree of the monomial α. Let $\tilde{\alpha}$ denote the constant-coefficient differential operator on $\mathbf{R}^d$ obtained from α by replacing the ith basis vector $e_i \in \mathbf{C}^d$ by $\partial/\partial x_i$, $i = 1, ..., d$. Then the Hermite polynomial $H_\alpha(x)$ corresponding to α is given by $(\tilde{\alpha} p_1)(x) = H_\alpha(-x)\, p_1(x)$ and the Hermite map $\alpha \mapsto H_\alpha(x)$ is a linear isometry of S into $L^2(\mathbf{R}^d, p_1)$ with dense range. If $\bar{S}$ is the completion of S in the above inner product, then the above map is a linear isometry of $\bar{S}$ onto $L^2(\mathbf{R}^d, p_1)$.

Recently [G1], [G2] L. Gross discovered a noncommutative version of the above result, with $\mathbf{R}^d$ replaced by a compact Lie group G. Here the analogue of the symmetric tensors are the elements of the universal enveloping algebra U of the corresponding Lie algebra $\mathfrak{g}$, while the analogue of the Gaussian is the heat kernel p_1 on G at time 1. What Gross discovered was the existence of a unique natural linear isometry between a completion of U and $L^2(G, p_1)$. Gross's proof [G1] is probabilistic and relies heavily on structures associated to a G-valued Brownian motion, in part motivated by analysis on path groups. Here we exhibit this isometry explicitly and we give a simplified proof.

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Statement of the Result

Most of the background material can be found in [S]. Below all function spaces are over $\mathbf{C}$. Let G denote a d-dimensional compact real Lie group and let $\langle\cdot,\cdot\rangle$ denote an Ad-invariant inner product on its Lie algebra $\mathfrak{g}$. We assume G is connected and simply connected. Let $\mathfrak{g}_c=\mathfrak{g}\otimes\mathbf{C}$ be the corresponding complex Lie algebra and let $\langle\cdot,\cdot\rangle_1$ be the unique complex inner product on $\mathfrak{g}_c$ extending $\langle\cdot,\cdot\rangle$.

For $n\geqslant 1$ let $\mathfrak{g}_c^{\otimes n}$ denote the n-fold tensor product (over $\mathbf{C}$) of $\mathfrak{g}_c$ with itself and let $\langle\cdot,\cdot\rangle_n$ denote the unique complex inner product on $\mathfrak{g}_c^{\otimes n}$ satisfying $\langle\xi_1\otimes\cdots\otimes\xi_n,\ \eta_1\otimes\cdots\otimes\eta_n\rangle_n=\prod_{i=1}^n\langle\xi_i,\eta_i\rangle_1$ for ξ_i, η_i, $i=1,\dots,n$, in $\mathfrak{g}_c$. For $n=0$ set $\mathfrak{g}_c^{\otimes 0}=\mathbf{C}$ with its usual complex inner product $\langle\cdot,\cdot\rangle_0$. Let $|\cdot|_n$ denote the corresponding norm, $n\geqslant 0$.

Let T denote the (algebraic) tensor algebra over $\mathfrak{g}_c$, i.e., all finite sums $\alpha=\sum\alpha_n$ with $\alpha_n\in\mathfrak{g}_c^{\otimes n}$, and for $h>0$ define a complex inner product on T by setting

$$\langle\alpha,\beta\rangle_h=\sum_{n=0}^{\infty}\frac{n!}{h^n}\langle\alpha_n,\beta_n\rangle_n,\qquad \alpha,\beta\in T,$$

where α_n, β_n, $n\geqslant 0$, are the homogeneous components of α and β respectively. Let T_h denote the completion of T in this inner product and let $\langle\cdot,\cdot\rangle_h$ and $|\cdot|_h$ denote the inner product and corresponding norm on the complex Hilbert space T_h.

For each $\xi\in\mathfrak{g}_c$ let $\tilde{\xi}$ denote the corresponding right-invariant vector field on G. Then the map $\xi\mapsto\tilde{\xi}$ is a Lie algebra homomorphism and hence extends uniquely to an associative algebra homomorphism $\alpha\mapsto\tilde{\alpha}$ of T onto the algebra $\mathbf{D}(G)$ of right-invariant differential operators on G. It follows from the Poincaré–Birkhoff–Witt theorem [S] that the kernel of this map is the two-sided ideal $J\subset T$ generated by elements of the form $\xi\otimes\eta-\eta\otimes\xi-[\xi,\eta]$, where $\xi,\eta\in\mathfrak{g}_c$, and hence the universal enveloping algebra $U=T/J$ is isomorphic to $\mathbf{D}(G)$. Let J_h denote the closure of J in T_h and let $U_h=J_h^{\perp}$ denote the orthogonal complement of J_h in T_h. It follows [Hi] from Theorem A below that the Hilbert space U_h is naturally isometric with the Banach completion of U in the quotient norm corresponding to $|\cdot|_h$.

Choose an orthonormal basis $\{\xi_1,\dots,\xi_d\}$ for $\mathfrak{g}$ and set $\Delta=\tilde{\xi}_1^2+\cdots+\tilde{\xi}_d^2$. Then the second-order right-invariant differential operator Δ is independent of the choice of basis, lies in the center of $\mathbf{D}(G)$, and generates a contraction semigroup $P_t=e^{t\Delta/2}$: $L^2(G)\to L^2(G)$, $t\geqslant 0$, where $L^p(G)=L^p(G,dx)$ and dx denotes Haar measure. Moreover there is a positive C^∞ kernel p_t such that P_t is given by convolution with p_t, $P_tf=p_t*f$, for $f\in L^2(G)$ and $t>0$.

Let $L^2(G, p_h)$ denote the complex Hilbert space of functions square-integrable against $p_h(x)\,dx$, $h > 0$.

THEOREM A. *Fix $h > 0$. Let $H(\alpha) = H_\alpha \in C^\infty(G)$ be given by*

$$(\tilde{\alpha} p_h)(x) = H_\alpha(x^{-1})\, p_h(x). \tag{1}$$

Then the map $H = H_h$: $T \to C^\infty(G)$ extends to a bounded linear transformation H_h: $T_h \to L^2(G, p_h)$ with kernel $J_h \subset T_h$. Moreover the extension is a linear isometry of U_h onto $L^2(G, p_h)$.

Let G_c denote the complexification of G. This is a complex Lie group naturally associated to G with Lie algebra $\mathfrak{g}_c$. Then [Ha] the heat kernel p_t extends uniquely to a holomorphic function on G_c. Let $\mathscr{H}$ denote the set of holomorphic functions on G_c.

The Hermite map H_h is one of a triad of isometries between U_h, $L^2(G, p_h)$ and $L^2(G_c, \mu_h) \cap \mathscr{H}$, where μ_h is a specific heat kernel on G_c, the other two being S_h and T_h:

$$\begin{tikzcd}
U_h \arrow[r, "H_h"] & L^2(G, p_h) \arrow[d, "S_h"] \\
L^2(G_c, \mu_h) \cap \mathscr{H} \arrow[u, "T_h"] \arrow[r, equal] & L^2(G_c, \mu_h) \cap \mathscr{H}
\end{tikzcd}$$

Here the Taylor map T_h associates to each holomorphic function in $L^2(G_c, \mu_h)$ its Taylor series at the identity (see below), while S_h associates to $f \in L^2(G, p_h)$ the unique holomorphic extension of $P_h f$ to G_c, $S_h f(z) = \int_G p_h(zx^{-1})\, f(x)\, dx$ for $z \in G_c$.

In the classical case $G = \mathbf{R}^d$, $G_c = \mathbf{C}^d$, the above commutative diagram was emphasized and developed by I. Segal, especially in the case $d = \infty$, see [BSZ]; in this context the map S_h is called the Segal–Bargmann transform. In the compact group case, the map S_h was defined and its isometry properties established by B. C. Hall [Ha], who used Fourier analysis on G.

By analogy with the development in [BSZ], $h > 0$ should be thought of as (proportional to) Planck's constant, $\sqrt{p_h}$ thought of as a ground state on configuration space G, and G_c should be thought of as phase space.

The existence of the Hermite isometry H_h: $U_h \to L^2(G, p_h)$ was discovered by L. Gross [G1]; his proof is infinite-dimensional in the sense it involves analysis on Wiener space. Subsequently the above explicit form and a simplified proof appeared in [Hi]. Nevertheless a portion of the proof in [Hi] relied on some aspects of the proof in [G1]. Later B. K. Driver [D] found a non-probabilistic proof of the isometry of the

Taylor map T_h, which, when combined with Hall's result, yielded a finite-dimensional "complex-variable" proof of Theorem A. Driver's ideas included varying Planck's constant $h > 0$. Here we present a "real-variable" proof of Theorem A which incorporates Driver's ideas directly into the top leg of the above diagram.

Proof of the Result

It turns out it is more convenient to work with the adjoint $K_h = H_h^*$ of H_h which goes from $L^2(G, p_h)$ to U_h. Throughout $h > 0$. Let $\{\xi_1, ..., \xi_d\}$ be an orthonormal basis for $\mathfrak{g}$. For $g \in C^\infty(G)$ and $n \geq 1$, set

$$\nabla^n g(x) = \sum_{i_1, ..., i_n = 1}^{d} \tilde{\xi}_{i_1} \cdots \tilde{\xi}_{i_n} g(x)\, \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}.$$

Also set $\nabla^0 g(x) = g(x)$. Then the maps $\nabla^n g\colon G \to \mathfrak{g}_c^{\otimes n}$, $n \geq 0$, do not depend on the choice of orthonormal basis.

For $g \in C^\infty(G)$ and $x \in G$, set

$$e^{h\nabla} g(x) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \nabla^n g(x),$$

whenever the series converges in T_h. The basic property of $e^{h\nabla}$ is

$$\langle e^{h\nabla} g(x), \alpha \rangle_h = \tilde{\alpha} g(x), \qquad \alpha \in T, \tag{2}$$

which can be easily verified on monomials. The Taylor map T_h: $L^2(G_c, \mu_h) \cap \mathscr{H} \to U_h$ appearing above is obtained by applying $e^{h\nabla}$ to the restriction to G of the holomorphic function and then evaluating at the identity. At the basis of Driver's proof of the isometry of T_h is the identity $\Delta |g|^2 = |\partial g|^2$ valid for holomorphic g. This identity also plays a crucial role in the proof of E. A. Carlen's "coherent-state" identity [C].

For any complex Hilbert space H let $L^2(G; H) = L^2(G, dx; H)$ denote the square-integrable H-valued functions on G with the usual inner-product.

Lemma B. *For each $f \in L^2(G)$, $e^{h\nabla} P_h f \in L^2(G; T_h)$ and $\|e^{h\nabla} P_h f\|_{L^2(G;\, T_h)} = \|f\|_{L^2(G)}$. Conversely, if $g \in C^\infty(G)$ and $\|e^{h\nabla} g\|_{L^2(G;\, T_h)} < \infty$, then there exists a unique $f \in L^2(G)$ satisfying $g = P_h f$.*

The proof is essentially integration by parts [Hi].

LEMMA C. *For* $f \in L^2(G)$, $e^{h\nabla}P_h f(x)$ *exists in* T_h *and*

$$|e^{h\nabla}P_h f(x)|_h^2 = \sum_{n=0}^{\infty} \frac{h^n}{n!} |\nabla^n P_h f(x)|_n^2 = P_h(|f|^2)(x), \tag{3}$$

for all $x \in G$.

To see this, set $u(x, h)$ equal to the infinite series and differentiate term-by-term to conclude that u satisfies the heat equation $\partial u/\partial h = \frac{1}{2}\Delta u$ on $G \times (0, \infty)$ with initial condition $|f|^2$. But this is the same as saying $u(x, h) = P_h(|f|^2)(x)$. The details are in [Hi].

Now define $K_h f = (e^{h\nabla}(P_h f))(e)$. Then by (2) K_h maps into U_h. Inserting $x = e$ in (3) establishes the isometry of K_h: $L^2(G, p_h) \to U_h$.

To establish the surjectivity of K_h, we will need a Lemma (Corollary F below) that is an immediate consequence of [G1, Lemma 8.2]. However here we present a direct Lie-theoretic proof that avoids any use of the path group ΩG as in [G1, Lemma 8.2]. That such a proof is possible was already suggested in [G1]. But first we recall some facts concerning the multiplicative structure in T.

For $\xi \in \mathfrak{g}_c$ set

$$e^{\otimes \xi} = \sum_{n=0}^{\infty} \frac{1}{n!} \xi \otimes \cdots \otimes \xi,$$

with n factors of ξ in the nth summand. Then $e^{\otimes \xi} \in T_h$ with $|e^{\otimes \xi}|_h = e^{|\xi|^2/2h}$. Now let $T_0 = \bigcap_{h>0} T_h$; then $e^{\otimes \xi} \in T_0$ for all $\xi \in \mathfrak{g}_c$.

The following Lemma [D], although completely elementary, will be extremely useful in what follows.

LEMMA D [D]. *If* $a > b + c$, $b > 0$, $c > 0$, *then* $|\alpha \otimes \beta|_a \leqslant C(a, b, c)\, |\alpha|_b\, |\beta|_c$ *for* α, β *in* T. *Hence* T_0 *is an associative algebra with unit* 1 *under* $\otimes$.

Now we recall the theory of the Hausdorff series [B, II§6]. Given two noncommuting indeterminates a, b, let $\mathrm{Alg}(a, b)$ denote the free associative algebra with unit generated by a, b. Given s, t in $\mathrm{Alg}(a, b)$ let $\wedge$ denote the wedge product $s \wedge t = st - ts$, and let $\mathrm{Lie}(a, b) \subset \mathrm{Alg}(a, b)$ denote the smallest linear subspace containing $1, a, b$ and closed under wedging with a, b on the left and on the right. $\mathrm{Lie}(a, b)$ is a Lie algebra under $\wedge$ and the elements of $\mathrm{Lie}(a, b)$ are *Lie polynomials*. Then there are homogeneous Lie polynomials $H_n(a, b)$ of degree n, $n \geqslant 1$, such that with $H(a, b) = H_1(a, b) + H_2(a, b) + \cdots$ one has an equality of formal power series

$$e^a e^b = e^{H(a, b)}, \tag{4}$$

in the sense that terms of like degree agree. Moreover the identity (4) is universal in the sense the projection of (4) holds in any particular associative algebra.

In particular if ξ, η are in $\mathfrak{g}$ then ξ, η are in the associative algebra $(T_0, \otimes)$. If $u \wedge v$ is interpreted as $u \otimes v - v \otimes u$ in T_0, then there is a Hausdorff series $H^{\otimes}(\xi, \eta) = H_1^{\otimes}(\xi, \eta) + H_2^{\otimes}(\xi, \eta) + \cdots$, with $H_n^{\otimes}(\xi, \eta) \in \mathfrak{g}_c^{\otimes n}$, $n \geqslant 1$, such that

$$e^{\otimes \xi} \otimes e^{\otimes \eta} = e^{\otimes H^{\otimes}(\xi, \eta)}; \tag{5}$$

here again equality means terms of like degree agree.

Now let $H_n(\xi, \eta)$, $n \geqslant 1$, be obtained from $H_n^{\otimes}(\xi, \eta)$, $n \geqslant 1$, by replacing $u \wedge v = u \otimes v - v \otimes u$ by $[u, v]$, the bracket in $\mathfrak{g}$. Since $H_n^{\otimes}(\xi, \eta)$ is a Lie polynomial, we obtain a polynomial map $H_n \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, for each $n \geqslant 1$. Then there is a symmetric neighborhood $\mathfrak{a}$ of the origin in $\mathfrak{g}$ such that the Hausdorff series

$$H(\xi, \eta) = \sum_{n=1}^{\infty} H_n(\xi, \eta),$$

converges to a real-analytic map $H \colon \mathfrak{a} \times \mathfrak{a} \to \mathfrak{g}$. Moreover if $\exp \colon \mathfrak{g} \to G$ denotes the exponential map, then $\exp \xi \exp \eta = \exp H(\xi, \eta)$ for ξ, η in $\mathfrak{a}$.

Let $\pi_h \colon T_h \to U_h$ denote the orthogonal projection and let $\mathbf{G}_h \subset U_h$ denote the elements of the form $\pi_h(e^{\otimes \xi})$, $\xi \in \mathfrak{g}$. We say a map $\mu \colon G \to U_h$ is *real-analytic* if $x \mapsto \langle \mu(x), \alpha \rangle_h$ is real-analytic in the usual sense, for all $\alpha \in U_h$.

The following lemma shows that a simply connected Lie group "lives" in (the completion of) the enveloping algebra of its Lie algebra. A simple modification shows that the complexification G_c also "lives" in U_h.

LEMMA E. *Under the map $(\alpha, \beta) \mapsto \pi_h(\alpha \otimes \beta)$, $\mathbf{G}_h$ is a group isomorphic to G. In fact there is a real-analytic group isomorphism $\mu_h \colon G \to \mathbf{G}_h$ satisfying*

$$\mu_h(\exp \xi) = \pi_h(e^{\otimes \xi}), \qquad \xi \in \mathfrak{g}. \tag{6}$$

Proof. By the very definition of J, we have $H_n(\xi, \eta) - H_n^{\otimes}(\xi, \eta) \in J$ for $n \geqslant 1$ and ξ, η in $\mathfrak{g}$. Hence

$$e^{\otimes \xi} \otimes e^{\otimes \eta} = e^{\otimes H(\xi, \eta)} \qquad \operatorname{mod} J_h$$

for ξ, η in $\mathfrak{a}$. Since all three exponential series converge in T_h and the product of exponentials is well-defined in T_h by Lemma D, this is equality in T_h and not just formal equality as in (5).

Let $\tilde{\mathbf{G}}_h$ denote the semigroup generated by the elements $\pi_h(e^{\otimes \xi})$, $\xi \in \mathfrak{a}$. By Lemma D $\tilde{\mathbf{G}}_h \subset U_h$ is well-defined. Since each $e^{\otimes \xi}$ is invertible in T_h,

$\xi \in \mathfrak{g}$, it follows that $\tilde{\mathbf{G}}_h$ is a group. By shrinking $\mathfrak{a}$, if necessary, we may assume that exp is a diffeomorphism of $\mathfrak{a}$ onto $A = \exp(\mathfrak{a})$. Define $\mu_h \colon A \to \tilde{\mathbf{G}}_h$ by setting $\mu_h(\exp \xi) = \pi_h(e^{\otimes \xi})$. Now choose $B \subset A$ a symmetric connected neighborhood of e satisfying $B^3 \subset A$ and set $\mathfrak{b} = \exp^{-1}(B) \subset \mathfrak{a}$. Since $\exp \xi \exp \eta = \exp H(\xi, \eta)$ for ξ, η in $\mathfrak{a}$, it follows that $\mu_h(xyz) = \pi_h(\mu_h(x) \otimes \mu_h(y) \otimes \mu_h(z))$ for x, y, z in B. Since G connected and simply connected, $\mu_h \restriction B$ extends to a group homomorphism $G \to \tilde{\mathbf{G}}_h$ [B, III§6.1].

Now for all or $\alpha \in U_h$,

$$\langle \alpha, \mu_h(\exp \xi_1 \exp \xi_2 \cdots \exp \xi_n) \rangle_h = \langle \alpha, e^{\otimes \xi_1} \otimes e^{\otimes \xi_2} \otimes \cdots \otimes e^{\otimes \xi_n} \rangle_h$$

for all $\xi_1, ..., \xi_n$ in $\mathfrak{b}$. Since $\bigcup_{n \geq 1} B^n = G$, it follows that $\mu_h \colon G \to U_h$ is real-analytic. Since (6) holds for $\xi \in \mathfrak{b}$ and both sides are real-analytic, (6) holds for all $\xi \in \mathfrak{g}$. It now follows that $\tilde{\mathbf{G}}_h = \mathbf{G}_h$ is a group and μ_h is surjective.

To establish injectivity of μ_h, let $f \colon G \to U(V)$ be a finite-dimensional unitary representation. Viewing f as a matrix-valued function on G and applying the heat semigroup P_h to f componentwise, we see [Hi] there is a self-adjoint $D \in End(V)$ such that $P_h f(x) = e^{hD} f(x)$ for all $x \in G$. From above we know the matrix entries of

$$K_h f = (e^{h\nabla} P_h f)(e) = (e^{h\nabla} e^{hD} f)(e) = e^{hD}(e^{h\nabla} f(e)) \in End(V)$$

are in U_h. Since e^{hD} is invertible, it follows that the matrix entries of $\psi_h \equiv e^{h\nabla} f(e)$ are in U_h. Now suppose $x = \exp \xi \in G$ and $\mu_h(x) = 1$, i.e., $e^{\otimes \xi} = 1 \bmod J_h$. Then

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} \frac{\xi^n f(e)}{n!} &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \xi^{\otimes n}, \psi_h \rangle_h \\ &= \langle e^{\otimes \xi}, \psi_h \rangle_h = \langle 1, \psi_h \rangle_h = f(1). \end{aligned}$$

(Here in writing down the convergent Taylor series we are recalling that f extends to an entire matrix-valued function on G_c.) Choosing f faithful completes the proof of injectivity and hence the lemma.

Corollary F [Gl, Lemma 8.2.]. *Fix $\alpha \in U_h$. Then there is a real-analytic $g = g_\alpha \colon G \to \mathbf{C}$ satisfying*

$$(\tilde{\beta} g)(x) = \langle \alpha, e^{\otimes \xi} \otimes \beta \rangle_h, \qquad x = \exp(-\xi),\ \xi \in \mathfrak{g},\ \beta \in T. \tag{7}$$

Proof. Set $g(x) = \langle \alpha, \mu_h(x^{-1}) \rangle_h$, $x \in G$. Then

$$\begin{aligned} &g(\exp(-\xi_n) \exp(-\xi_{n-1}) \cdots \exp(-\xi_1)\, x) \\ &\quad = \langle \alpha, e^{\otimes \xi} \otimes e^{\otimes \xi_1} \otimes \cdots \otimes e^{\otimes \xi_n} \rangle_h, \end{aligned} \tag{8}$$

for $\xi_1, ..., \xi_n \in \mathfrak{g}$ and $x = \exp(-\xi) \in G$.

Replacing ξ_n by $t\xi_n$ in (8) and differentiating at $t=0$ yields

$$\tilde{\xi}_n g(\exp(-\xi_{n-1})\cdots\exp(-\xi_1)\,x) = \langle \alpha, e^{\otimes \xi} \otimes e^{\otimes \xi_1} \otimes \cdots \otimes e^{\otimes \xi_{n-1}} \otimes \xi_n \rangle_h. \tag{9}$$

Replacing ξ_{n-1} by $t\xi_{n-1}$ in (9) and differentiating at $t=0$ yields

$$\tilde{\xi}_{n-1}\tilde{\xi}_n g(\exp(-\xi_{n-2})\cdots\exp(-\xi_1)\,x) = \langle \alpha, e^{\otimes \xi} \otimes e^{\otimes \xi_1} \otimes \cdots \otimes e^{\otimes \xi_{n-2}} \otimes \xi_{n-1} \otimes \xi_n \rangle_h.$$

Continuing in this manner we obtain (7). This completes the proof of the corollary.

Now fix an arbitrary $\alpha \in U_h$; by inserting $\xi=0$ in (7), we obtain $\tilde{\beta}g(e) = \langle \alpha, \beta \rangle_h$ or $e^{h\nabla}g(e)=\alpha$; thus to complete the proof of isometry of K_h we need to show $g=P_hf$ for some $f\in L^2(G)$ for then $\alpha = e^{h\nabla}g(e) = e^{h\nabla}P_hf(e)=K_hf$.

To this end by Lemma B it is enough to show $x \mapsto |e^{h\nabla}g(x)|_h$ is in $L^2(G)$. Instead, by analogy with Driver's idea in the holomorphic setting, we show first $x \mapsto |e^{(h-2\varepsilon)\nabla}g(x)|_{h-2\varepsilon}$ is in $L^\infty(G)$ for $\varepsilon>0$ small. For this we use Lemma D and (7): For $\beta \in T$ and $x=\exp(-\xi)$,

$$\begin{aligned} &|\langle e^{(h-2\varepsilon)\nabla}g(x), \beta\rangle_{h-2\varepsilon}| \\ &\quad = |\tilde{\beta}g(x)| = |\langle \alpha, e^{\otimes \xi}\otimes \beta\rangle_h| \leqslant |\alpha|_h\, |e^{\otimes \xi}\otimes\beta|_h \\ &\quad \leqslant C_\varepsilon\, |\alpha|_h\, |\beta|_{h-2\varepsilon}\, |e^{\otimes \xi}|_\varepsilon = C_\varepsilon\, |\alpha|_h\, |\beta|_{h-2\varepsilon}\, e^{|\xi|^2/2\varepsilon} \\ &\quad \leqslant C'_\varepsilon\, |\alpha|_h\, |\beta|_{h-2\varepsilon}, \end{aligned} \tag{10}$$

where we used the fact that G is the image of a bounded subset of $\mathfrak{g}$ under exp. Since (10) is true for all $\beta\in T$, we obtain $|e^{(h-2\varepsilon)\nabla}g(x)|_{h-2\varepsilon} \leqslant C'_\varepsilon\,|\alpha|_h$. Hence by Lemma B for each $h/2>\varepsilon>0$ we can find f_ε satisfying $g=P_{h-2\varepsilon}f_\varepsilon$ which by Lemma C satisfies

$$\begin{aligned} \|f_\varepsilon\|^2_{L^2(G,\,p_{h-2\varepsilon})} &= \sum_{n=0}^{\infty} \frac{(h-2\varepsilon)^n}{n!}\,|\nabla^n g(e)|_n^2 \\ &\leqslant \sum_{n=0}^{\infty} \frac{h^n}{n!}\,|\nabla^n g(e)|_n^2 = |e^{h\nabla}g(e)|_h^2 = |\alpha|_h^2. \end{aligned}$$

Since $\{p_{h-2\varepsilon} : 0<\varepsilon<h/4\}$ is uniformly bounded away from zero we see $\{f_\varepsilon : 0<\varepsilon<h/4\}$ is bounded in $L^2(G)$. Hence $f=\lim_{\varepsilon\to 0} f_\varepsilon$ exists weakly in $L^2(G)$ which yields $g=\lim_{\varepsilon\to 0} P_{h-2\varepsilon}f_\varepsilon = P_hf$ weakly in $L^2(G)$. Since this implies $K_hf = e^{h\nabla}P_hf(e) = e^{h\nabla}g(e)=\alpha$, this concludes the proof of surjectivity hence the proof of isometry of K_h. Since $H_h = K_h^* = K_h^{-1}$ is straightforward to verify [Hi], this completes the proof of Theorem A.

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